

Models and Logical Consequence *

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Abstract

This paper deals with the adequacy of the model-theoretic definition of logical consequence. Logical consequence is commonly described as a necessary relation that can be determined by the form of the sentences involved. In this paper, necessity is assumed to be a metaphysical notion, and formality is viewed as a means to avoid dealing with complex metaphysical questions in logical investigations. Logical terms are an essential part of the form of sentences and thus have a crucial role in determining logical consequence.

Gila Sher and Stewart Shapiro each propose a formal criterion for logical terms within a model-theoretic framework, based on the idea of invariance under isomorphism. The two criteria are formally equivalent, and thus we have a common ground for evaluating and comparing Sher and Shapiro philosophical justification of their criteria. It is argued that Shapiro *blended approach*, by which models represent possible worlds under interpretations of the language, is preferable to Sher's *formal-structural view*, according to which models represent formal structures. The advantages and disadvantages of both views' reliance on isomorphism are discussed.

1 Introduction

There is no dispute that Tarski's definition of logical consequence is one of the landmarks of logic in the 20th century. The model-theoretic

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definition Tarski devised has such a strong intuitive appeal, that for many philosophers and mathematicians today model-theoretic consequence just is logical consequence. Indeed, Tarski's accomplishment is not one of those earth-shattering surprising mathematical results; it has an air of obviousness, at least in retrospect – which is a main source of its strength.

Tarski intended to capture, with his model-theoretic definition, a pre-existing concept of consequence, in a manner as faithful as possible to its ordinary usage. Tarski analysed the ordinary concept of consequence as *necessary* and *formal* – and those are the features that he was committed to capturing. So, despite the general acceptance of Tarski's definition, there is still room to ask whether, or how, it captures the main features of logical consequence.

The philosophical question of the basis of the model-theoretic definition of consequence, in its bluntest and most critical form, was formulated by John Etchemendy [4, 5]. Etchemendy argued that Tarski's definition is both conceptually and extensionally defective. Though Etchemendy's attack was not received as detrimental to the model-theoretic project, it posed a two pronged challenge to its adherents. One part of the challenge was to defend Tarski by refuting Etchemendy's interpretation, and another was to argue for the adequacy of model theory for defining consequence, regardless of the specifics of Tarski's work.

The main focus of this paper will be the second part of Etchemendy's challenge. I will address the issue of supporting the model-theoretic definition through the works of two main proponents of the semantic tradition: Gila Sher and Stewart Shapiro. Different emphases in their work may have caused the impression that they are not contributors to the same debate, but I will show that the contrary is true: they are in fact dealing with the same issues and have much in common.

Two threads intertwine throughout this paper. One is mathematical, and the other philosophical. The theorists we deal with share a common philosophical framework, which consists of an analysis of the concept of logical consequence as necessary and formal, and a method for showing that the mathematical apparatus adequately captures this concept. Their common method for showing the adequacy of model theory is to take models to represent something in the non-mathematical realm. The point of comparison here will be the sort of items that models represent, and this is where Sher and Shapiro

diverge. Sher takes models to represent *formally possible structures*, whereas Shapiro takes them to represent *possible worlds under interpretations of the nonlogical terms of the language*. I argue that Shapiro's philosophical account is more successful: it captures necessity and formality in a straightforward manner which is faithful to the conceptual common ground.

The common mathematical framework will be a first order language with model-theoretic semantics. Within this framework I compare different ways of fixing the logical terms of the language. Both Sher and Shapiro provide mathematical criteria for logical terms which are essential to their accounts of formality. In the course of the paper, I point out (and prove in the appendix) that their formal criteria for logical terms are equivalent, and this draws their approaches even closer. Their common mathematical framework allows a close comparison of their philosophical views.

The plan of the paper is as follows. In section 2, I formulate and discuss the conditions of *Necessity* and *Formality* on logical consequence, in a way, I think, that would be agreeable to the philosophers I discuss. I refer to the notion of the *form* of a sentence, but do not provide a full analysis of it. I attribute to the form of a sentence a certain role: referring to it allows us to argue validly, while avoiding intricate metaphysical questions. This minimalistic characterization of form is compatible with the views I discuss, and will suffice for the evaluation of various accounts of model-theoretic consequence. In section 3, model theory enters the picture. I present Shapiro's *logic-as-model* approach that sets the ground for the connection between the mathematical apparatus of model theory and the philosophical notion of logical consequence. In section 4, I map the different possible approaches within the *logic-as-model* framework, among which are Sher and Shapiro's views. I go through some disadvantages in Sher's view that Shapiro overcomes. In section 5, I point out, based on [6], that Shapiro and Sher's mathematical characterizations of logical terms are equivalent, and through an additional mathematical result, I substantiate the claim in favor of Shapiro's philosophical account. Finally, following some concluding remarks in section 6, the appendix in section 7 contains an outline of the proof of the equivalence result.

2 Necessity and Formality

I shall focus here on a fairly conventional understanding of logical consequence as *necessary* and *formal*. I will make only minimal assumptions on what necessity and formality amount to, and will not offer a full explication of these notions. In my assumptions, I follow by and large Tarski's outline of the intuitive concept of logical consequence [12, p. 414]. This is by no means intended to be an interpretation of Tarski's idea, but just a common ground to start from, general enough to be consistent with many of the formulations. Let an *argument* be a pair consisting of a set of sentences (the premises) and a sentence (the conclusion).¹ Inspired by Tarski, we can characterize logical consequence by two conditions:

Necessity: In a logically valid argument, the conclusion follows necessarily from the premises, or, equivalently, it is not possible for all premises to be true and the conclusion false.

Formality: The logical validity of an argument is determined by its form. Specifically, an argument is valid if and only if every other argument with the same form is valid.

Indeed, the two authors I focus on take on board these conditions. For Sher, “The intuitive notion of logical consequence is that of necessary and formal consequence” [10, p. 668]. Sher has an elaborate conception of formality that goes beyond the condition of *Formality* I formulated above, but that Tarskian condition is certainly acceptable by her. Formality, she would add, is primarily attributed to properties of objects in the world, and only derivatively to items in a language, as the latter denote the former.

Shapiro (1998) lists nine different intuitive characterizations of logical consequence of modal, epistemic and linguistic nature, among which are: “It is not possible for the members of Γ to be true and yet Φ false” and “The truth of the members of Γ guarantees the truth of Φ in virtue of the forms of the sentences (or propositions)” [7, p. 132].

I take the notion of *necessity*, as it is used above, to be a meta-

¹For the purpose of this paper I take the *relata* of logical consequence to be sentences. Other accounts are not excluded, as long as they involve sentences (e.g. sentence-context of utterance pairs).

physical notion, broadly understood, which may be explained in terms of possible worlds. I refer to possible worlds throughout the paper, but do not assume a specific doctrine with respect to their nature. Specifically, I remain neutral with respect to realism or antirealism regarding possible worlds. My only assumption is that *necessity* can be understood as *truth in all possible worlds*.

However construed, possible worlds involve heavyweight questions regarding, e.g., the nature of existence, properties, objecthood, etc. But logical practice, it seems, attempts to satisfy *Necessity* without being completely steeped in metaphysics. It seems that logic is a discipline that tries to capture metaphysical necessities in a systematic way that avoids contentious metaphysical questions as much as possible. This is where formality enters the picture.

Note that *Formality* assumes that arguments have forms, and moreover, that each argument has a unique form. In this we follow a well-entrenched tradition in logic, though, I submit, not all characterizations of the formality of logic pertain to forms of sentences.² Yet, we are not told what *form* is. The answer to this question is far from obvious, and it requires further assumptions beyond the conditions *Necessity* and *Formality*. At this point, I do not want to commit myself as to what exactly constitutes form.

Considering the different approaches of the writers we are dealing with, we can come up with a minimal characterization of form, a common denominator for this debate. I will hereforth address the issue of formality through a certain *role* that is commonly attributed to form, whatever form may be. The emphasis will be on this role, rather than a full and explicit definition or characterization of form. Although I try to make minimal assumptions about form and I presume that the role I attribute to it is quite appealing, I do not expect it to be in agreement with *all* prevalent views. I do contend that the idea I present is consonant with the approaches of Sher and Shapiro, the two main authors I discuss.

To the point: one feature which we might expect of form is that it will permit us to bypass metaphysical considerations to some extent. Whatever form is, it should relieve us from having to take sides in thorny metaphysical disputes or even from deciding which of them are

²Presupposing that sentences have *unique* forms might be contested as well, but we will do so nonetheless to keep things simple.

coherent. So, for instance, we would like ‘Sarah likes cheese’ to follow logically from ‘Everyone likes cheese’, regardless of the metaphysical status of Sarah (Does she necessarily exist? Does she exist at all? Could she have had different parents than she actually has?) and that of cheese (is it a natural kind? Could it have grown on trees?). By this some necessarily true sentences, such as perhaps [*insert here your favourite metaphysical doctrine*], will not be logically valid. A step towards this desirable feature of form is to have forms of arguments be subject to mathematical handling, as they are in conventional formal systems of logic. A formal system might not capture all arguments satisfying *Necessity*, but only a restricted class of necessary arguments, which the system can detect on basis of their form. This is not to say that there are no other reasons to treat forms mathematically, but that treating forms of arguments as mathematical objects is in line with the desired feature.

Surely, formal systems for logical consequence may be involved with metaphysical questions to some extent. I do not deny any connection between metaphysics and logic – on the contrary: I take the condition of *Necessity* to be the primary condition on logical consequence. Yet we would like as much metaphysics as possible to be out of the way. It is more a matter of the *quantity* of metaphysical questions that form helps to reduce. We can require that the involvement of formal systems with metaphysics will be controlled and systematized at the outset, and the contentious metaphysical questions be reduced to a minimum. Moreover, we expect that even if some assumptions are endorsed at the outset and embedded in the system, the practice of using the system to determine the validity of specific arguments be metaphysically innocuous: the verdict of the system on a given case where it applies will not be subject to metaphysical dispute.

I attribute to form a specific role, that of making it possible to capture *Necessity* without getting involved in metaphysics. Some might view the suggested role of formality as additional or as an outcome of *form*’s more basic features (e.g. its generality, topic-neutrality etc.)³ Or, this role could also be taken as *the* basic feature of form. In any case, what is important for my current concerns is that a good account of form will be such where metaphysical questions are largely kept out of the way.

³Sher is probably a good example. See section 4.3 for Sher’s view of formality.

Despite their different treatments of formality, Sher and Shapiro seem to agree that logic should capture *Necessity* without immersing itself in metaphysics. As we shall see, this has significant implications for their philosophical approach to form. Neither Sher nor Shapiro specify the role of formality as I did, but this role can be inferred from their views nonetheless. We shall go into the details of their views in subsequent sections, but the outline of the interpretive inference is as follows: The metaphysical approach to semantics (see section 4.1) is problematic according to both Sher and Shapiro. The main reason it is problematic are the metaphysical questions it entails ([10, p. 661], [7, p. 149]). Incorporating an appropriate account of form (or, of logical terms, which constitute forms of sentences) will get rid of the problematic metaphysical questions ([10, p. 672], [7, p. 152]). So, by both Sher and Shapiro, in a proper account of logical consequence, form fulfills the role we attributed to it – of avoiding metaphysical questions.

3 Models and Logical Consequence

Contemporary logical practice uses mathematical techniques to determine the validity of arguments on the basis of their form. Model theory allows a rigorous, precise treatment of validity, and moreover, provides a promise to keep logic out of a metaphysical mess. Let us assume that the language we are dealing with is of first-order, the details of which we leave open for the moment. A *model* is a pair $\langle D, I \rangle$ such that D , the domain, is a non-empty set, and I is a standard interpretation function. According to Tarski’s model-theoretic definition of logical consequence, the sentence φ *follows logically* from the sentences of the set Γ (or: the argument $\langle \Gamma, \varphi \rangle$ is *logically valid*) if and only if φ is true in each model where all sentences in Γ are true.⁴

In order for the model-theoretic definition to be adequate, the relation captured by it has to satisfy *Necessity* and *Formality*. To Tarski it was obvious that his definition “agrees quite well with common usage” [12, p. 417], but he did not show that it indeed does so.⁵

⁴I am employing here “model” in the contemporary usage, not the original Tarskian one, as my concern is the philosophical underpinnings of the model theory we currently use. The differences will not concern us.

⁵Tarski only points out that the desirable features of consequence are outcomes

The model-theoretic tradition has more to say about form than I have so far. It is a common assumption in the works of Tarski and his followers, and in contemporary logic in general, that the key factor determining the form of a sentence is its logical vocabulary. The question of form is thus usually dealt with through the investigation of logical terms. Tarski left the choice of logical terms as a (somewhat) free parameter in his definition of logical consequence (though he didn't think everything goes⁶). Later logicians tried to fix this parameter and define logical terms. Nonetheless, whatever the logical terms are taken to be, the model-theoretic definition is designed to satisfy *Formality*: truth-preservation over models is determined by the chosen logical vocabulary. Although I have previously not committed to what forms (or logical terms) are, I have attributed to them the role of helping us avoid metaphysical questions. I will consider this feature of form as a test for a choice of logical terms later on.

Apparently, for Tarski, the adequacy of the model-theoretic definition was obvious. But if we don't agree about the obviousness of these issues, we may articulate two philosophical gaps in the Tarskian account. Tarski's model-theoretic definition still leaves open how it:

- (1) captures necessity, and
- (2) makes it possible to avoid metaphysical questions (for some choice of logical terms).

Before we go on to examine ways by which these gaps may be filled, some issues that will not be dealt with should be set aside. Problematic metaphysical assumptions, one might claim, arise in model theory even before addressing the question of logical terms. For instance, the use of sets might be disputed. Models are set-theoretic entities, and as such, are subject to queries pertaining to the nature and existence of sets. Such worries may drive us to abandon set theory in favor of better foundations for model theory, or abandon models altogether in favour of, say, a proof-theoretic account. I will not attempt to deal

of his definition. "In particular," he writes, "it can be proved, on the basis of this definition, that every consequence of true sentences must be true, and also that the consequence relation which holds between given sentences is completely independent of the sense of the extra-logical constants which occur in these sentences" (*ibid*). It should be noted that it is not all that clear that Tarski adheres to *Necessity* as we formulated it, see [14, pp. 25-29]. Indeed, had Tarski provided a proof of the adequacy of his definition, it might have been clearer what he meant by the word "must".

⁶[12, p. 418].

here with metaphysical worries about sets, and just assume sets are metaphysically innocent.

There have been two prominent attitudes to the gaps in the Tarskian account. One is of viewing them as marking insurmountable difficulties in the Tarskian project [3, 4]. The alternative, positive attitude, of trying to fill the philosophical gaps is adopted by contemporary adherents of model-theoretic consequence such as Sher and Shapiro.⁷

Now how would one go about showing that a formalism is philosophically adequate? A common ploy is to tie the formalism to the object of inquiry through *representation*. Models, apart from their role as mathematical objects, correspond to something in the non-mathematical world that has to do with logical consequence, and in this way we ensure *Necessity*. So, for instance, we could take each model to represent a possible world, and straightforwardly capture *Necessity*. In what follows we will see that this first attempt will not do, but a modification of it will be more successful.

Not all writers who take models to represent something state what such a relation requires. We can refer here to Shapiro, who provides us with a framework where the requirements on representation are made explicit. Shapiro presents the view of *logic-as-model* by which certain features of the mathematical apparatus we use in logic, the *representors*, represent features of logical consequence in natural language ([7], see also [2]). It is important to note our double use of the word “model”. The “model” in “logic-as-model” does not refer to the models of model theory, but to the role of the whole formal system. I use the word “model” in both senses, assuming that context can disambiguate.⁸ By the view of *logic-as-model*, a formal system as a whole serves as a *mathematical model* for logical consequence. A model in general simplifies and idealizes the modeled phenomenon, aiming at an easier to handle, yet adequate, account. What it takes for

⁷To be precise, Etchemendy thinks that the conceptual adequacy of model theory can be salvaged, but as he stresses, it would be in a way very foreign to Tarski’s so called reductive analysis. In fact, Etchemendy’s approach to model theory is quite close to that of Shapiro, but he fiercely criticizes the attribution of such an approach to Tarski, see [5].

⁸Obviously, there is a connection between the two uses of “model”. But when referring to the models of model theory we use “model” as a technical, mathematical, notion, where the aspect of models of model theory modeling something will be attributed to them only in virtue of being part of a bigger model, which consists of the whole formal system.

a model to be adequate is not always clear-cut, and it depends on the purpose of the investigation. Shapiro explains that “typically, there is no question of ‘getting it exactly right’ ” and that “there should be a balance between simplicity and closeness of fit” [7, p. 138]. In Shapiro’s view, logical consequence is a cluster notion that involves varying intuitions. He claims that model-theoretic consequence can serve as a good mathematical model of the notion of logical consequence based on modal and semantic intuitions, which we articulated in the conditions of *Necessity* and *Formality* [7, p. 148]. The models of model theory (“models” in the other sense) are merely items in the formal system that are nominees to be representors or artifacts in the big model.

Sher too connects the formal system to the concept of logical consequence through representation, though she does not employ an elaborate account such as Shapiro’s logic-as-model. Sher seems to demand, however, a closer match between representors and what they represent, and this, we will see, poses a disadvantage to her account.

4 What Models Represent

Taking on the logic-as-model approach, it is natural to assume that the models of model theory are not mere *artifacts* of the system but are *representors*. I will work under that assumption, and the guiding question will be: what do models represent? The following subsections present four answers to this question. I first discuss in brief two unsatisfactory initial attempts at a solution, inspired by Etchemendy (1990), though I will follow Sher’s presentation and terminology. The shortcomings of the first proposed answers lead to alternative solutions by Sher and Shapiro, which will be our main focus.

4.1 Metaphysical Semantics

A way of understanding the condition of *Necessity* is as a requirement of truth preservation in all possible worlds, while model-theoretic consequence preserves truth in all models – which suggests correlating models to possible worlds. In *metaphysical semantics* (or “representational semantics” in Etchemendy’s terms), models represent possible worlds, or possible configurations of the world (see [4, ch. 2]).

The metaphysical approach does not fit common practice very well, and would entail a revision of standard model theory (see [10, p. 659], [7, p. 143]). It is therefore most appropriate to view metaphysical semantics as not merely a philosophical perspective on model theory, but as an approach on how the apparatus of model theory should be adjusted and employed, regardless of current practice. Metaphysical semantics requires a fully interpreted language, so that evaluating sentences at possible worlds would be possible. In this approach, a sentence is true in a model if and only if the model represents a possible world where the sentence is true (or the proposition expressed by the sentence holds).⁹ A sentence such as ‘The board is red all over and green all over’ (or its formalization) would be true in some model only if it is metaphysically possible for the board to be both red and green all over. That is, this sentence would be rendered logically false.

It is easy to see that metaphysical semantics, if viable, satisfies *Necessity*. To the extent that all possible worlds can be represented by models in the system, the model-theoretic consequence relation would be necessarily truth-preserving. Metaphysical semantics simply equates logical consequence with *necessary* consequence. In this approach, logically valid arguments just are those that are necessarily truth-preserving. This would not be agreeable to those who think that *Formality* is a substantive condition on logical consequence (whatever forms may be).

From the point of view of our current concerns: if we can use the model-theoretic apparatus for metaphysical semantics, albeit with some revision, we can take care of the gap posed in (1) above, i.e. capture necessity. However, (2) – avoiding metaphysical questions – is completely ignored in this approach. Sher lays out the main disadvantage of metaphysical semantics:

[M]etaphysical semantics requires solutions to the most obscure and thorny questions of general metaphysics. [... W]e come upon recalcitrant questions of identity, essential properties, moral and rational agency, meaning, etc. that have baffled philosophers for years. [10, pp. 660-661]

⁹We assume here, together with Sher, Shapiro and Etchemendy that there is a natural, straightforward way of evaluating sentences in possible worlds or configurations of the world. For a critical discussion of this assumption see [1].

Sher's concern here is that of avoiding metaphysical questions. Whatever forms of sentences might be in this approach, they do not fulfill the role we have attributed to them: metaphysical semantics (as its name already discloses) does not save us from a single metaphysical question. In order to construct our semantics, we need to know the modal status of every sentence in our language (see also [4, p. 25]).

These considerations suffice for us to move on and examine other possible approaches to semantics. Clearly, metaphysical semantics is not adequate at filling the philosophical gaps of the model-theoretic definition of consequence.

4.2 Linguistic Semantics

Another option for the model-theorist is to follow the *linguistic semantics* approach, and view the variations in models as representing variations in *interpretations* of the language.¹⁰ This, again, is an approach that uses the apparatus of models with some modifications to contemporary standard practice – and so cannot be viewed as a perspective on ordinary model theory.¹¹ The domain of discourse in this view is held fixed across models. Linguistic semantics frees itself of much of the metaphysical burden posed by metaphysical semantics, and is concerned only with the world as it *is*, not as it *could have been*, and thus each model represents the actual world. The relation between language and the world, the different ways language can be interpreted – that is what makes the difference between models.

We can use an example by Shapiro to display the difference between linguistic and metaphysical semantics. Consider the sentence 'Snow is black' (or a formal rendering of it). In the metaphysical approach to semantics, a model in which this sentence is true represents a possible world where the color of snow is black. In the linguistic approach, a model in which the sentence is true represents an interpretation of the language in which the sentence *means* something true, e.g. one where 'black' denotes *white* [7, p. 142].

¹⁰The idea of linguistic semantics is inspired by Etchemendy's "interpretational semantics". But by contrast, in interpretational semantics models do not represent anything, they *are* interpretations. Etchemendy attributes interpretational semantics to Tarski. The difference between *linguistic* and *interpretational* semantics is not crucial for our purposes and specifically, we are not concerned with Tarski exegesis.

¹¹See [10, p. 666] and [7, pp. 143, 149].

In linguistic semantics, all re-interpretations of the language must respect *form*. On the assumption that logical terms determine the forms of sentences,¹² those terms keep their meanings across models – they are “held fixed”. So what varies from model to model is just the interpretation of the *nonlogical* terms. In the example above, ‘black’ can be re-interpreted as it was only if it is considered as a nonlogical term. The linguistic perspective leans heavily on the distinction between logical and nonlogical terms. As a result, it is much more faithful to *Formality*.

Linguistic semantics does not suit standard model theory, since the latter does not keep the domain fixed. But suppose we made the proper adjustments to our model theory so it will fit the linguistic perspective. That is, fix the domain to be a set that represents the actual world. Have we made any progress with respect to the gaps we mentioned in (1) and (2)? It seems that with linguistic semantics the tables have turned: we are now doing better at avoiding metaphysical questions. We need be only concerned with metaphysical questions pertaining to one possible world, the actual one, rather than all possible worlds. By contrast, satisfying *Necessity* appears to be the weak link. The idea motivating linguistic semantics might be that modality could be reduced to semantic variability. Perhaps, with the right choice of logical terms, dealing only with the actual world would suffice – variations in the world would be accounted for through variations in the language. This hope, however, cannot be sustained.

We will not go over all the possible choices of logical terms, but will note that linguistic semantics is not adequate with the standard first order selection of logical terms. Consider the following sentence φ : $\forall x \exists y Rxy \wedge \forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz) \wedge \forall x \neg Rxx$, which states that the binary relation R is serial, transitive, and irreflexive [4, p. 118]. Such a relation is possible only in an infinite domain. Now assume that the universe consists of only finitely many objects, so the fixed domain for our semantics is finite. In that case, $\neg \varphi$ would be true in all models, that is, it would turn out to be logically true (R is the only nonlogical term in the sentence). This would be in violation of *Necessity*, if we countenance infinite possible worlds. We could then argue that the world is infinite. That would change the logical status

¹²To be precise, what determines the form of a sentence is not only logical terms and their arrangement in the sentence, but also the grammatical categories of nonlogical terms and their iterations in the sentence.

of $\neg\varphi$ to not logically true (and not logically false). In this case, $\neg\varphi$ would not be a counter-example to *Necessity*. But then, arguably, we are letting our logic rely on a substantial metaphysical assumption regarding the size of the world, in contrast to the spirit of *Formality*. Even though we might not be able to avoid all metaphysical questions, this one might involve a price too heavy to pay.¹³

The prospects of finding a choice of logical terms which would both satisfy *Necessity* and not involve weighty metaphysical assumptions are bleak. Narrowing down to just the actual world does not free us from metaphysical investigations. Some questions about what *actually is* are already heavily loaded and intricate.¹⁴ Thus, linguistic semantics does not deliver the adequacy of model-theoretic consequence. Nevertheless, as we learn from Sher and Shapiro, metaphysical and linguistic semantics do not exhaust the possible approaches model-theorists can employ.

4.3 The Formal-Structural View

A natural solution to the inadequacy of each of Etchemendy's proposed semantic views is to combine the two so that the features of each of them compensates for the disadvantages of either one or the other. Such is Shapiro's *blended approach*, according to which models represent possible worlds under interpretations of the nonlogical terms of the language. *Necessity* and *Formality* are captured in this approach, as they are captured separately in representational and interpretational semantics respectively.¹⁵ The blended approach is discussed extensively in the next subsection.

Sher seems to be aware of the possibility of combining metaphysical and linguistic semantics, but she chooses a different route – her

¹³Etchemendy proposes a modification that compensates for the fixed domain. The quantifiers are analyzed as having two components, so that, e.g., \exists is understood as *some-thing*, where *some* is fixed and *thing* varies over subsets of the domain. Note that as subsets of the domain never have cardinality greater than that of the domain, even with this modification the logical truth of $\neg\varphi$ depends on finitude of the world, see [4, pp. 112-120].

¹⁴Indeed, metaphysical investigations are a way to learn things about the actual world, sometimes more feasible than empirical methods.

¹⁵This obvious solution is absent from Etchemendy's presentation of the two approaches to semantics in [4]. His own approach to semantics, which he describes in [5] as a version of representational semantics, is, however, very similar to Shapiro's.

formal-structural view. For Sher, “a (logical) model represents a formally possible structure of objects relative to the primitive terms of a given language” [10, p. 675]. These structures consist of objects and correspond in some way to possible states of affairs, but they are emphatically not possible worlds. Indeed, Sher is an adamant critic of the view of models as representing possible worlds, as was previously pointed out in the discussion of metaphysical semantics. She argues that any position that takes models as representing possible worlds entails formidable difficulties; it imports metaphysical conundrums into logical practice, and requires taking into account an unreasonable amount of information for any reasonably rich language. The exceedingly complex metaphysical issues that arise in the description of possible worlds cannot be expected to be solved in the formal system, and therefore cannot serve as a basis for model-theoretic semantics. This worry is not limited to metaphysical semantics, in which logical consequence is equated with necessary consequence, and in which each model is viewed as representing a possible world; Sher’s critique is also aimed at views involving possible worlds that take the category of logical consequence as merely derived from the category of necessary consequences.¹⁶ We see that Sher requires of formality to fulfill the role we attributed to it at the start: drawing the set of formal and necessary consequences from the set of necessary consequences will not give a good account of logical consequence if all the metaphysical questions remain.

Yet how can Sher present models as representing or corresponding somehow to possible states of affairs without viewing them as representing possible worlds (or possible ways the world might be, or any other equivalent metaphysical notion)? In the course of the present subsection I suggest that Sher’s view of models as formally possible structures is unsatisfactory, and that her view inevitably collapses to the blended approach, which does incorporate possible worlds.

According to Sher, and in line with the presuppositions of this work, logical consequences are the formal and necessary consequences, and in Sher’s words “formal necessity is a particular case of both necessity and formality” [10, p. 668]. The concept of *formality* is a

¹⁶Sher implies that a semantic approach which takes logical consequence as a subset of necessary consequence has two options: either it shares the disadvantages of metaphysical semantics [10, p. 659], or it is simply useless, “tantamount to giving an account of the distinctive nature of *logical* semantics” [10, p. 661].

key notion in Sher's approach, and she tackles it through the definition of logical terms.

For Sher, "Logical terms are formal in the sense of denoting properties and relations that are, roughly, intuitively structural or mathematical" [10, p. 668]. Sher is a follower of the Tarskian tradition, according to which the logical terminology that appears in a sentence determines its form. Like Tarski, she holds that as a result of *Formality*, logical consequence is independent of our ability to distinguish between objects in the world. In Tarski's words:

Moreover, since we are concerned here with the concept of logical, i.e. *formal*, consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds, this relation cannot be influenced in any way by empirical knowledge, and in particular by knowledge of the objects to which the sentence X or the sentences of the class K refer. The consequence relation cannot be affected by replacing the designations of the objects referred to in these sentences by the designations of any other objects. [12, pp. 414-415]

Sher [9, 10] proposes a mathematical definition of logical terms. Formally, Sher's system of *universal logic* extends standard classical predicate logic. Sher concentrates on variations of first order logic; in her system, each logical term is a truth-functional connective, or an n -place predicate or functor of level 1 or 2. The general criterion for formality in Sher's approach (also known as the Tarski-Sher Approach) is:

(F) A term is formal iff it is invariant under isomorphic structures. ([10, p. 677], see also [13])

The terminology satisfying (F) includes standard logical terminology, as well as non-standard logical terms, such as *finite* and *most*. Truth functional connectives can be incorporated in different ways,¹⁷ but to keep things simple, Sher just defines them to be logical. Thus, we have *Sher's criterion for logical terms*:

¹⁷Truth functional connectives are invariant under isomorphic propositional structures, and can fit the schema of Sher's definition of logicality. Alternatively, the connectives can be viewed as generalized quantifiers (see [11]).

C is a logical term (in universal first order logic) if it is a truth-functional connective, or a predicate or functional expression of level 1 or 2 that satisfies (F) (see [10, p. 679]).

It has been less noted that models also have a central role in Sher's approach to formality. Models in Sher's account are representors – they represent *formally possible structures*. Now let us take a step back. The basic conditions at the basis of our discussion are *Necessity* and *Formality*. We have agreed (i.e. all writers referred to here agree) that possible worlds are an admissible way of speaking of necessity. At this point we are also assuming (the widely held assumption, no doubt) that logical terms determine the form of sentences. Let us look at whether the notion of *formally possible structure* is a reasonable addition to our conceptual apparatus, and whether it can ground the adequacy of the model-theoretic definition.

A straightforward way to obtain all logical consequences is to start out from the class of all necessary consequences (where the conclusion follows necessarily from the premises). Logical consequences are a subset of this class: they are the necessary consequences that are also “formal”. In this method one might rely on a possible-worlds-oriented semantics, as in metaphysical semantics, and in this way obtain all necessary consequences. From those, all logical consequences can be obtained by using an additional criterion, such as: a logical consequence is a necessary consequence that is also necessary under every re-interpretation of the nonlogical terms. This is, as noted before, Shapiro's approach, that will be discussed in the next subsection.

Sher's approach is, at least ostensibly, completely different. She contends that the straightforward approach entails all the problems of incorporating possible worlds into the logical system. However, she suggests that the collection of arguments that satisfy necessity can be narrowed down in a different manner – by looking at a wider notion of possibility. The range of possible worlds gives a certain collection of necessary consequences, the ones for which there is no possible world where the premises are true and the conclusion false. A wider range of states or possibilities would bring about a narrower collection of arguments, which would be included in the one obtained by the original range.

Sher thus extends the standard concept of *possibility*. She avoids

using *possibility* as that which accompanies the concept of *necessity* and is ordinarily explicated through possible worlds, dealing instead with what she terms *formal possibility*, which accompanies the concept of *formal necessity*, and can include cases that are not metaphysically possible. Thus, for example, there is a formal possibility in which the statement ‘The board is red all over and green all over’ is true. Each formal possibility is represented by a model:

[T]he totality of models reflects “possibilities” that in general metaphysics might be ruled out by nonformal considerations. That is to say, the notion of *possibility* underlying the choice of models is wider than in metaphysics proper. [9, p. 138]

In this manner Sher avoids the metaphysical discussion of possible worlds; her only obligation is to characterize formally possible structures. Formally possible structures are represented by models, which are set-theoretic constructs. In this way, Sher reduces formal possibility to mathematical objects:

The reductive approach to modal notions is based on the idea that imprecise intuitive notions can be reduced to clear yet adequately strong precise notions... On my conception, formal possibility is reduced to mathematical existence and formal necessity to mathematical generality: “It is formally possible that Φ ” to “There exists at least one mathematical (set theoretical) structure \mathfrak{S} such that Φ holds in \mathfrak{S} ” and “It is formally necessary that Φ ” to “For all mathematical (set theoretical) structures \mathfrak{S} , Φ holds in \mathfrak{S} .” [10, p. 682]

Let us take a closer look at Sher’s manner of reasoning. The concept of logical consequence revolves on the concept of necessity and on the concept of formality. Logical consequences are the formally necessary consequences; there is therefore no need to deal with the general concept of necessity, but only with the formally necessary – what is formally necessary is, obviously, both formal and necessary. And one might think that in the same manner in which the formally necessary can be established, so can the dual concept of the formally possible. But here we meet a problem. Whereas the extension of the formally

necessary is included in that of the ordinary concept of necessity, the extension of the formally possible *extends* that of the ordinary concept of possibility, and it not clear how it is to be attained. It's obvious that to get the formally necessary we intersect the formal with the necessary. It is less clear how to extend the concept of possibility: Sher does not tell us how to come up with the formal possibilities which are not metaphysical possibilities.

By employing the concept of formal possibility, Sher seemingly gains the advantage over views that involve possible worlds, incorporating necessity into her system without the “metaphysical baggage”. But when adopting the new concepts of formal necessity and formal possibility, she abandons the familiar ground of accepted concepts. *Formal possibility*, however, is not the result of the intersection of two pre-given concepts, but of the expansion of one of them, and cannot be straightforwardly derived from what is *formal* and what is *possible*. That is to say, the concept of formal possibility is a completely new modal concept which Sher employs as her proposed theoretical basis for model-theoretic semantics. Sher does not give a substantive definition of *formal possibility* – that which models represent in her approach. Nor does Sher adopt it as a primitive concept. As we shall see, an attempt to achieve a clear understanding of this concept leads to the conclusion that *formal possibility* can best be understood as *interpretation of the nonlogical terms in a possible world (or state of affairs)*, despite Sher's obvious wish to leave out possible worlds.

Consider the example sentence ‘The board is red all over and green all over’. Under ordinary modal terminology this sentence is “impossible”: it is false in all possible worlds. But this sentence would be, by Sher's analysis, formally possible. The meaning of this is that there is a formally possible structure for the language (in this case, natural language or a fragment thereof) in which this sentence is true, and this structure is represented by a model, in which the corresponding formalized sentence is true, for:

Intuitively, a (logical) model represents a formally possible structure of objects relative to the primitive terms of a given language. [10, p. 675]

But what is a “formally possible structure relative to the primitive terms of a given language”?

According to Sher, models represent something that has to do with possible states of affairs:

Tarski's semantics... [uses] a semantic apparatus which allows us to represent the relationship between language and the world in a way that distinguishes formal and necessary features of reality. The main semantic tool is the model, whose role is to represent possible states of affairs relative to a given language. [9, p. 138]

[W]e would like to know how models represent all intuitive possibilities with respect to a given language: what features of possible states of affairs correlate with what features of models, how differences between models suffice to represent all relevant differences between states of affairs. [10, p. 657]

If we combine this with Sher's stance that nonlogical terms are re-interpreted in every model, we may venture to summarize her position thus: models represent constructs that correlate to possible states of affairs, and nonlogical terms are re-interpreted in them. My claim is that is simply another way for saying that a formally possible structure is, in fact, an interpretation of the nonlogical terms in a possible world (or state of affairs).¹⁸ The sentence 'The board is red all over and green all over' is true in, for example, a formally possible structure where the domain is $\{a\}$, 'the board' denotes a , and 'red all over' and 'green all over' both denote $\{a\}$. And this structure corresponds to an interpretation of the nonlogical terms in a possible world where there is just one object, and in which the extensions of 'red all over' and 'green all over' consist of the only object in the world.

Sher attempts to use formally possible constructions to circumvent possible worlds, or any metaphysical import for that matter. Sher's circumvention is only superficial: the most plausible way to understand Sher's use of formally possible constructions is as possible worlds under interpretations of the nonlogical terminology. Thus, the alternative Sher offers to metaphysical and to linguistic semantics is

¹⁸The difference between possible worlds and states of affairs (the former committing to maximality not assumed by the latter) should not concern us here; most of the metaphysical questions alluded to section 4.1 with respect to possible worlds will re-arise when using possible states of affairs.

ultimately a combination of both. However, a view of models as representing possible worlds under interpretations does not necessarily entail the metaphysical difficulties Sher attributes to it. This point will be elaborated in the next subsection.

At the current stage my claim is the following: when one tries to understand the new modal concept Sher employs, and endow it with substance in a way that complies with her remarks, this concept collapses to the ordinary concepts of necessity and formality that are ordinarily explicated through possible worlds and interpretations respectively. My criticism still leaves one loose end: it seems that Sher succeeds in establishing a theoretical basis to logic, while steering clear of the difficulties arising from taking models to represent possible worlds. As a reply to my criticism, Sher might claim that if her account avoids metaphysical issues, there is a marked difference between it and any account in which possible worlds are indeed incorporated. Thus, the introduction of formal possibilities might have some beneficial feature I have overlooked. In the next subsection I show that despite Sher's contention, there is no need to throw away possible worlds in order to steer clear of metaphysical questions.

4.4 The Blended Approach

Let us now look more closely into Shapiro's suggestion to "blend" metaphysical and linguistic semantics. Shapiro contends that logical consequence is a cluster-notion which involves modal, semantic and epistemic intuitions. *Necessity* and *Formality* are good representatives of the modal and semantic intuitions, and together they are a basis for a *conglomeration* notion of consequence, hereafter referred to as *Cong.*:

Cong.: Φ is a logical consequence of Γ if Φ holds in all possibilities under every interpretation of the nonlogical terminology in which Γ holds. [7, p. 148]

Model-theoretic consequence, according to Shapiro, can be used as a good mathematical model of *Cong.* We simply take models to represent possible worlds under interpretations of the nonlogical terms. This blend of metaphysical and linguistic semantics is not as artificial as it may seem. Shapiro advises us to be sceptical of the metaphysical-linguistic dichotomy in the first place. "Most philosophers", he says,

“do not doubt that there is a difference between the role of language and the role of the world in determining the truth-values of sentences, but many hold that this distinction is not at all sharp” [7, p. 147]. Take a model M where ‘Snow is black’ is true, and assume that both ‘snow’ and ‘black’ are nonlogical. In the blended approach it is indistinguishable whether M represents a way the world could be in which snow is black, or an interpretation of the language by which ‘Snow is black’ turns out to be true. It is sufficient that we know all possibilities are accounted for, and in general that we have a good model of logical consequence.

Here, though, Sher’s concerns reappear. Obviously, both *Necessity* and *Formality* are satisfied in the blended approach. What is less obvious is how the gap previously mentioned as (2) is addressed: It seems that we need to answer a multitude of metaphysical questions in order to represent all possible worlds, before we add the additional dimension of varying the language and blur the *truth in virtue of the world/truth in virtue of language* distinction. But this contention is wrong. There is no need to take two separate steps, one to account for all metaphysical possibilities and another for possible interpretations. We can make use here of the logic-as-model approach: we don’t need to account for all possible worlds, as long as *Necessity* is made sure to be satisfied. We need not be worried whether it is possible for snow to be black. Varying the interpretations of ‘snow’ and ‘black’ with a minimal variation of the domain will account for all possibilities with respect to ‘Snow is black’.

The blended approach resembles linguistic semantics when re-interpretations step in instead of possible worlds. The difference is that the blended approach does not abjure possible worlds: it does not claim that modality can be completely reduced to semantic variation.

Let us make this idea more precise. We need our models to be such that varying interpretations can take care of all possibilities while avoiding as much metaphysics as possible. Learning from the failure of linguistic semantics, we let the size of the domain vary, taking account of all the possible sizes the world could have. The question of the sizes of possible worlds involves us with some metaphysics, but as we shall see, this is the only metaphysical question we have to deal with.

Shapiro suggests the following property for “any model theory worthy of the name” [7, p. 152]:

The isomorphism property. A language L (in the formalism) is said to have the isomorphism property iff for every formula φ and models M and M' for L with corresponding assignments s and s' such that $\langle M, s \rangle$ and $\langle M', s' \rangle$ are isomorphic with respect to the non-logical terms in φ , $\langle M, s \rangle \models \varphi$ if and only if $\langle M', s' \rangle \models \varphi$ [7, p. 151].¹⁹

As Shapiro explains, “The isomorphism property is a manifestation of the intuition that logical truth and logical consequence should be a matter of ‘form’, to the extent that isomorphism preserves ‘formal’ features of various models (whatever those are)” [7, pp. 151-152].

The isomorphism property can be viewed as a constraint on logical terms. In fact, as we’ll see in the next section, it is equivalent to Sher’s criterion for logical terms. Shapiro, however, takes the isomorphism property to be a necessary, though perhaps not sufficient condition on formal languages *vis-à-vis* their logical terms. How does this constraint help us in addressing (2)?

Shapiro acknowledges that logic should avoid metaphysical issues, at least to some extent (see [7, p. 149]). The isomorphism property, Shapiro shows, has the following corollary:

Corollary. Let L be a first order language. Let D and D' be two domains of the same cardinality, and assume that the isomorphism property holds for L . Then a sentence φ is true under every interpretation of the nonlogical vocabulary of L in D if and only if it is true under every interpretation of the nonlogical vocabulary of L in D' . (See [7, p. 152])

The corollary teaches us that given the isomorphism property, it is enough to take one domain of each size and all the model-theoretic interpretations on it to account for all possibilities model theory can account for. So we do not have to represent each and every possible world by a separate model. Each model will represent all possible worlds of its size, under some interpretation of the nonlogical vocabulary. *Necessity* will still be satisfied, and we arrive at a good position with respect to metaphysical questions: the only information we need

¹⁹In his [7], Shapiro does not mention assignments, but it is natural to add them in, since a formula is evaluated by a model only under an assignment. Alternatively, we may restrict formulas to sentences, as Shapiro does in [8]. This difference doesn’t have significant implications for our purposes.

regarding possible worlds is the class of cardinalities of all possible worlds, that is, we just need to know the sizes possible worlds may have.

Standard first order logic achieves an even better result. From the downward Löwenheim-Skolem theorem we can further conclude that a countable domain suffices to account for all infinite possible worlds. (Each finite possible world, if there are any, must still be accounted for by a finite model of the same cardinality). This may provide reason to restrict the class of logical terms even further, and require that terms would be admitted as logical only if the Löwenheim-Skolem theorems hold for the resultant logic. This, however, is not a mild restriction: it would rule out a great many terms that the isomorphism property allows. Generalized quantifiers such as *there exist uncountably many* and many other terms that have to do with cardinality do not agree with the Löwenheim-Skolem theorems (not to mention second-order logic).²⁰

Even when the isomorphism property is the only constraint on logical constants, the blended approach succeeds in both capturing the required kind of necessity and in avoiding *most* of the complex metaphysical issues it involves. This indeed captures the gist of the role we set for formality: the selected logical terminology makes it possible to ground the necessary truth-preservation of a restricted class of arguments using (mostly) formal means, without having to deal with an overwhelming burden of metaphysical questions.

5 Tying the threads

A comparison of Sher and Shapiro's semantic approaches reveals that Shapiro's is more satisfactory at filling the philosophical gaps in Tarski's account of consequence. It is more straightforward: *Necessity* is captured without resorting to new, unexplained modal notions. One could of course take *formal possibility* and *formal necessity* as the primitive notions underlying logical consequence, but Sher starts with the plain notions of metaphysical possibility and necessity as we do. She then employs the idea of a *formally possible structure* whose role

²⁰If we further require countable compactness, the resultant logic will not be stronger than first order logic by Lindström's theorem, and the isomorphism property will be almost devoid of content, as will Sher's criterion.

is that of avoiding the metaphysics involved in incorporating possible worlds. As long as this notion remains primitive or unexplained, it is not clear that it captures *Necessity*. We have seen that upon scrutiny, this notion collapses to that of *possible world under interpretation of the nonlogical terms*.

Shapiro and Sher do not share exactly the same conceptual apparatus, as they differ in their treatment of the notion of *formality*. A thorough analysis of *formality* leads Sher to her invariance criterion for logical terms. Shapiro, on the other hand, proposes the isomorphism property as a necessary condition on languages. Interestingly, though, Sher and Shapiro's conceptions of formality are not all that different, at least on the technical side. When we bring their criteria to a common framework, namely first order languages (by Sher's restriction), they turn out to be equivalent.

Equivalence result. Let L be a first order language with logical terms of level 1 or 2 (truth-functional connectives, predicates and functional expressions), and countably many nonlogical predicates from each arity. We assume also that the model theory is a standard classical one. Then the isomorphism property holds for L iff the predicates and functional expressions which are logical terms in L are invariant under isomorphisms.^{21,22}

As shown by the equivalence result, the choice between Sher's criterion and Shapiro's isomorphism property becomes a practical matter depending on the specific task at hand. The isomorphism property has the advantage of conciseness and generality. On the other hand, if we have a language L for which we already know that the logical terms are well behaved in Sher and Shapiro's terms, Sher's criterion makes it easier to introduce an additional logical term to L .

Recall that the corollary to the isomorphism property shows that metaphysical questions can be reduced in systems satisfying the isomorphism property. By virtue of the equivalence result, the corollary to the isomorphism property also follows from Sher's criterion for logical constants. Hence the evasion of metaphysical questions can be

²¹Note that our use of the term *logical term* is neutral and does not presuppose any criterion for admissibility.

²²See the appendix for an outline of the proof, which appears in full in [6].

assumed in Sher's system, even when models are taken to represent possible worlds. Not every aspect of a possible world needs to be represented while the desired extension of logical consequence can still be achieved. Sher doesn't have to concern herself with the contentious notion of *formal structure* or *formal possibility*.

6 Conclusion

The formality of logic is commonly described as independence from a certain kind of information or content. For Sher, it is independence from the identity of particular objects. Shapiro aims to have formal languages that do not involve us in intricate questions on modality. I haven't offered in this paper a precise type of content which logic should disregard. I used the term "metaphysical questions" to refer, quite broadly, to the type of questions logic tries to avoid. The attitude I aimed to capture, shared by prominent writers in the area, is one by which logical consequence encompasses all possibilities (satisfies *Necessity*) without getting entangled in all that these possibilities involve. We would like valid arguments to preserve truth no matter what, in any case, but we are very far from clear on what all cases are. We then seek a mathematical apparatus that captures valid arguments without relying on contested issues.

There is no consensus on what the formality of logic amounts to. This is reflected in recent debates on the issue of logical terms. Sher and Shapiro approach the problem from quite different angles, but I believe that their accounts are both motivated to some extent by the attitude I posed. Both of them propose formal systems that are purported to capture *Necessity* without getting entangled in metaphysics. Shapiro is most successful in showing how this can be done – through the *logic-as-model* perspective combined with viewing models as representing possible worlds under interpretations of the nonlogical vocabulary. The latter principle ensures *Necessity*, the former relieves from the need to represent all possible worlds.

Sher and Shapiro (and many others) deal with formality through forms of sentences, and more specifically, through logical terms. Logical terms have gained such centrality in the philosophy of logic, so that the question of the formality of logic often turns to one of the formality of some *terms*. Adopting the attitude I delineated here,

however, does not commit one to there being a *category* of formal terms. Some characterizations of logical terms are simply more beneficial than others in keeping logic formal and away from problematic metaphysics. Sher's invariance criterion and Shapiro's isomorphism property suit our general purpose.

To be sure, I have not claimed that metaphysical questions can or should be avoided *completely*. What seems to fit the proposed picture of logic is a more pragmatic approach, by which we try to balance the poles of *Necessity* and *Formality*. When devising formal systems, we seek a mathematical apparatus which will compensate for our less than satisfactory grasp of the notion of *necessity* involved in logic. This aim can be achieved in various ways, and does not require a search for the one "correct" extension of logical terms or consequence in general.

7 Appendix

Equivalence result. Let L be a first order language with logical terms of level 1 or 2 (truth-functional connectives, predicates and functional expressions), and countably many nonlogical predicates from each arity. We assume also that the model theory is a standard classical one. Then the isomorphism property holds for L , with the class of models for L , if and only if the predicates and functional expressions which are logical terms in L are invariant under isomorphisms.

Proof Outline. The left to right direction (from the isomorphism property to Sher's criterion) is proved by constructing, for each logical term C , a formula ϕ where C is the only logical term, and then using the isomorphism property on ϕ . We show this only for the simple case of first-level predicates.

Let C be an n -place first-level predicate for some $n \geq 1$, and assume that C is a logical term in L . We need to show that it satisfies Sher's criterion, i.e. that it is invariant under isomorphic structures. Sher assigns to each logical term C a function f_C such that for each model M , $f_C(M)$ is C 's extension in M . Let M and M' be models with domains A and A' , and $\langle b_1, \dots, b_n \rangle \in A^n$, $\langle b'_1, \dots, b'_n \rangle \in A'^n$ such that the structures $\langle A, \langle b_1, \dots, b_n \rangle \rangle$ and $\langle A', \langle b'_1, \dots, b'_n \rangle \rangle$ are isomorphic. We need to show that $\langle b_1, \dots, b_n \rangle \in f_C(M)$ iff $\langle b'_1, \dots, b'_n \rangle \in$

$f_C(M')$. We now define assignments s and s' for M and M' respectively, thus:

$$s(x_i) = \begin{cases} b_i & \text{if } 1 \leq i \leq n \\ b_n & \text{if } i > n \end{cases} \quad s'(x_i) = \begin{cases} b'_i & \text{if } 1 \leq i \leq n \\ b'_n & \text{if } i > n \end{cases}$$

Note that the structures $\langle A, \langle b_1, \dots, b_n \rangle, s \rangle$ and $\langle A', \langle b'_1, \dots, b'_n \rangle, s' \rangle$ are isomorphic.

Now consider the formula $\varphi = C(x_1, \dots, x_n)$. From the previous claim we get that $\langle M, s \rangle$ and $\langle M', s' \rangle$ are isomorphic with respect to the nonlogical terms in φ (the nonlogical terms in φ include only variables²³). So by the assumption that the isomorphism property holds, $\langle M, s \rangle \models \varphi$ iff $\langle M', s' \rangle \models \varphi$. Therefore, by the definitions of s and s' , $M \models C[b_1, \dots, b_n]$ iff $M' \models C[b'_1, \dots, b'_n]$, thus $\langle b_1, \dots, b_n \rangle \in f_C(M)$ iff $\langle b'_1, \dots, b'_n \rangle \in f_C(M')$, as required.

The right to left direction is proved by induction, the details of which we leave to the reader.²⁴

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²³More precisely: the nonlogical terms in φ include *at most* variables. Variables can be considered as either nonlogical terms, or as neither logical nor nonlogical - which is how Sher treats them, see [9, p. 84]. That decision does not affect our proof.

²⁴For more details of the proof of the equivalence result, see [6].

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